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B. Sc. Part III (Hons) 5th Paper

METRIC SPACES (CONTD.)

Q: If d is a metric for X, prove that the function $d^*: X \times X \to R$ defined as

$$d^*(x,y) = \frac{d(x,y)}{1+d(x,y)}$$ is also a metric for X.

Soln We have to show that d^* is a metric on X. For this, we prove all the conditions for being a metric.

1. We prove that, $d^*(x,y) \geq 0 \ \forall \ x,y \in X$.

$\because$ d is a metric $\Rightarrow d(x,y) \geq 0$

$\Rightarrow 1+d(x,y) \geq 0$

$$\Rightarrow \frac{d(x,y)}{1+d(x,y)} \geq 0$$

$\Rightarrow d^*(x,y) \geq 0$. proved

2. We prove that $d^*(x,y) = 0 \Leftrightarrow x = y$.

$\because$ d is a metric $\Rightarrow d(x,y) = 0 \Leftrightarrow x = y$ ——(1)

Now let $d^*(x,y) = 0 \Leftrightarrow \dfrac{d(x,y)}{1+d(x,y)} = 0$

$\Leftrightarrow d(x,y) = 0$

$\Rightarrow d^*(x,y) = 0 \Leftrightarrow x = y$ [using (1)]

Proved

3. we prove that $d^*(x,y) = d^*(y,x)$.

$\because$ d is a metric $\Rightarrow d(x,y) = d(y,x)$

Now, $d^*(x,y) = \dfrac{d(x,y)}{1+d(x,y)} = \dfrac{d(y,x)}{1+d(y,x)}$

$\Rightarrow d^*(x,y) = d^*(y,x)$ Proved

4. we show that $d^*(x,y) \leq d^*(x,z) + d^*(z,y)$

$$d^*(x,z) = \frac{d(x,z)}{1+d(x,z)}$$

$$\Rightarrow d^*(x,z) \geq \frac{d(x,z)}{1+d(x,z)+d(z,y)} \quad \text{——— (2)}$$

because $1+d(x,z)+d(z,y) \geq 1+d(x,z)$

$$\Rightarrow \frac{1}{1+d(x,z)} \geq \frac{1}{1+d(x,z)+d(z,y)}$$

$$\Rightarrow \frac{d(x,z)}{1+d(x,z)} \geq \frac{d(x,z)}{1+d(x,z)+d(z,y)}$$

Similarly $d^*(z,y) \geq \dfrac{d(z,y)}{1+d(x,z)+d(z,y)}$ ——— (3)

Adding (2) and (3)

$$d^*(x,z) + d^*(z,y) \geq \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)}$$

$$\Rightarrow d^*(x,z) + d^*(z,y) \geq \frac{d(x,y)}{1+d(x,y)} = d^*(x,y)$$

$$\Rightarrow d^*(x,y) \leq d^*(x,z) + d^*(z,y)$$ Proved

Hence d^* is a metric on X.